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Real Spin-Clifford Bundle and the Spinor Structure of Space-Time

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By analyzing the conditions for the existence on a space-time $\mathscr{L}$ of a global algebraic spinor field, we prove the following result, known as Geroch's theorem: A necessary and sufficient condition for $\mathscr{L}$ to admit a spinor structure is that the orthonormal frame bundle $F_0(\mathscr{L})$ have a global section. Our proof, which does not use in any stage the complexification of $\mathbb{R}_{1,3}$ (the space-time Clifford algebra), is simple, requiring only the explicit construction of the algebraic spinor and the spinorial metric within $\mathbb{R}_{1,3}$ and elementary facts about associated bundles and the bundle reduction process. This is to be compared with the original proof, which uses the full algebraic topology machinery. We also clarify the relation of the covariant spinor structure and Graf's e-spinor structure.

1. INTRODUCTION

Definition 0. Let M be a four-dimensional, real, Hausdorff, connected, paracompact manifold. Let TM (T^*M) be the tangent (cotangent) bundle. A Lorentzian manifold is a pair (M, g), where $g \in \sec T^*M \times T^*M$ is a Lorentz metric of signature (p, q) with $p=1$ (or 3) and $g=3$ (or 1). A space-time $\mathscr{L}$ is a triple (M, g, ∇), where (M, g) is a noncompact, time-oriented and space-time-oriented Lorentzian manifold and ∇ is the Levi-Civita connection of g in (M, g). For what follows we choose without loss of generality $(p, q)=(1, 3)$. This point will be further discussed below. The principal fiber bundle (PFB) associated to $T\mathscr{L}$ (or $T^*\mathscr{L}$) is π: $P_{O(1,3)}(\mathscr{L}) \to \mathscr{L}$ and is equivalent to $F(\mathscr{L})$, the frame bundle. By $F_0(\mathscr{L}) = P_{SO_+(1,3)}(\mathscr{L})$ we denote the bundle of oriented Lorentz tetrads.

Definition 1 (Bichteler, 1968). A covariant spinor structure (CSS) for $F_0(\mathscr{L})$ consists of a PFB $\pi_s: P_{Spin_+(1,3)}(\mathscr{L}) \to \mathscr{L}$ with group $SL(2, \mathbb{C}) = Spin_+(1,3)$ [$\simeq Spin_+(3,1)$] and a map

$$s: \quad P_{Spin_+(1,3)} \to F_0(\mathscr{L})$$

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satisfying the following conditions:

(i) $\pi(s(p)) = \pi_s(p)$; $\forall p \in P_{Spin_+(1,3)}(\mathscr{L})$.
(ii) $s(pu) = s(u)\Lambda(u)$; $\forall p \in P_{Spin_+(1,3)}$ and Λ: $SL(2, \mathbb{C}) \to SO_+(1, 3)$ is the double covering of $SO_+(1, 3)$ by $SL(2, \mathbb{C})$.

It is well known that a necessary and sufficient condition for the existence of the spinor structure is the vanishing of the second Stiefel-Whitney class of $\mathscr{L}$, i.e., $w_2(\mathscr{L}) = 0$. For Riemannian manifolds this result was proved first by Milnor (1963) and for Lorentzian manifolds by Bichteler (1968). See also Crumeyrole (1969), Lee (1973), Popovici (1970), and Bugajska (1979).

It is quite clear from Definition 1 that if $F_0(\mathscr{L})$ is trivial, then a CSS always exists. The main objective of the present paper is to prove the converse, i.e., to prove the theorem stated in the abstract.

Definition 2. A spinor bundle for $\mathscr{L}$ is a vector bundle of the form

$$\bar{S}(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_{\bar{\rho}} V$$

where $\bar{\rho}$: $Spin_+(1, 3) \to GL(V)$ is a representation of $SL(2, \mathbb{C})$ on some vector space V.

Definition 3. A c-spinor field of type (ρ, V) on $\mathscr{L}$ is a section of $\bar{S}(\mathscr{L})$.

Alternatively, we can also define a c-spinor field of type (ρ, V) as an equivariant map ψ: $P_{Spin_+(1,3)}(\mathscr{L}) \to V$ such that

$$\psi(pu) = \rho(u^{-1})\psi(p), \qquad \forall p \in P_{Spin_+(1,3)}(\mathscr{L}), \qquad \forall u \in SL(2, \mathbb{C})$$

Definition 4. The (undotted) two-component c-spinor field (Weyl spinor field) corresponds to $V = \mathbb{C}^2$ (the two-dimensional complex space) and ρ the $D^{(1/2,0)}$ representation of $SL(2, \mathbb{C}) = Spin_+(1, 3)$ (Figueiredo *et al.*, 1990).

Now, in Figueiredo *et al.* (1990) we study in detail how to represent all types of covariant spinors (c-spinors) and corresponding spinorial metrics used by physicists (Pauli c-spinors, Dirac c-spinors, two-component undotted and dotted c-spinors) within the Clifford algebras $\mathbb{R}_{3,0}$ (the Pauli algebra), $\mathbb{R}_{1,3}$ (the space-time algebra), $\mathbb{R}_{3,1}$ (the Majorana algebra), and $\mathbb{R}_{4,1}$ (the Dirac-algebra) [for details of the notation see Figueiredo *et al.* (1990)]. The object that represents a given c-spinor in a given Clifford algebra we call an algebraic spinor (a-spinor). Algebraic spinors do not need in general to be elements of a minimal left (or right) ideal in the appropriate Clifford algebra.

Indeed, Dirac c-spinors are represented in $\mathbb{R}_{3,1}$ by Clifford numbers that do not belong to a minimal ideal (Figueiredo *et al.*, 1990). However, as shown in detail in Figueiredo *et al.* (1990), all c-spinor fields of physical interest can be represented as elements of a minimal left (or right) ideal in $\mathbb{R}_{3,0}$ ($\simeq \mathbb{R}^+_{1,3}$) and $\mathbb{R}_{1,3}$ ($\mathbb{R}^+_{1,3}$ denotes the even subalgebra of $\mathbb{R}_{1,3}$).

In particular, given the orthonormal frame $\{E_0, E_1, E_2, E_3\}$ of $\mathbb{R}^{1,3}$ (Minkowski space), the (undotted) two-component c-spinors can be represented by the elements of the ideal $I = \mathbb{R}^+_{1,3} e$, where $e = \frac{1}{2}(1 + E_3 E_0)$ is simultaneously a primitive idempotent of $\mathbb{R}_{3,0} \simeq \mathbb{R}^+_{1,3}$ and $\mathbb{R}_{1,3}$.

All this suggests the following.

Definition 5. A real spinor bundle of $\mathscr{L}$ is a bundle

$$S(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_\rho \mathbf{M}$$

where $\mathbf{M}$ is a left module for $\mathbb{R}_{1,3}$ or $\mathbb{R}^+_{1,3}$ and ρ: $Spin_+(1,3) \to SO_+(1,3)$ is the representation given by left multiplication by elements of $Spin_+(1,3) \subset \mathbb{R}^+_{1,3}$ (Figueiredo *et al.*, 1990; Blaine Lawson and Michelsohn, 1983).

It is quite clear from the construction of $I = \mathbb{R}^+_{1,3} e$ that when $\mathbf{M} = I$, $S(\mathscr{L})$ represents $\bar{S}(\mathscr{L})$ with ρ the $D^{(1/2,0)}$ representation of $SL(2, \mathbb{C})$. We have the following statement.

Definition 6. An algebraic spinor field (ASF) is a section of $S(\mathscr{L})$.

Definition 7. A complex spinor bundle for $\mathscr{L}$ is a bundle

$$S_C(\mathscr{L}) = P_{Spin_+(1,3)} \times_\rho \mathbf{M}_C$$

where ρ: $Spin_+(1,3) \to SO_+(1,3)$ is the representation given by left multiplication by elements of $Spin_+(1,3) = \mathbb{R}^+_{1,3}$ and $\mathbf{M}_C$ is a complex left module for $\mathbb{R}_{1,3} \otimes \mathbb{C} = \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$.

Now, given that $\mathbb{R}_{1,3}$ can be considered as a module over itself by left multiplication l, we are naturally led to the definition of the following real spinor bundle, which we call the real spinor-Clifford bundle.

Definition 8. $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_\iota \mathbb{R}_{1,3}$.

It is quite clear that $Cl_{Spin_+(1,3)}(\mathscr{L})$ is a "principal $\mathbb{R}_{1,3}$ bundle," i.e., it admits a free action of $\mathbb{R}^+_{1,3}$ on the right. $P_{Spin_+(1,3)}(\mathscr{L}) \to \mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ is natural embedding which comes from the natural embedding $Spin_+(p, q) = \{u \in \mathbb{R}^+_{p,q} \mid \tilde{u}u = 1\}$ valid for $p + q \leq 5$ (Figueiredo *et al.*, 1990). From the above it is quite clear that the properties of the real spinor bundles for $\mathscr{L}$ can be studied in $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$, since, in particular, dotted and undotted two-component spinor fields and the Dirac c-spinor fields can be represented as appropriate ideal sections of $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$. We shall show explicitly in Section 2 that from the condition for the existence of a global two-component a-spinor field in $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ a proof of Geroch's theorem results. It is important to remark that $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ is not the so-called Clifford bundle (Popovici, 1976; Figueiredo *et al.*, 1990; Graf, 1978; Blau, 1987). Indeed, the Clifford bundle $\mathcal{C}\ell(\mathcal{L})$, which always exists independently of the existence or not of the CSS, is given by the following statement.

Definition 9. $\mathcal{C}\ell(\mathcal{L}) = P_{SO_+(1,3)}(\mathcal{L}) \times_{\rho_c} \mathbb{R}_{1,3}$, where

$$\rho_c: \quad SO_+(1,3) \to Aut(\mathbb{R}_{1,3})$$

Now, if we remember that there exists a representation

$$Ad: \quad Spin_+(1,3) \to Aut(\mathbb{R}_{1,3})$$

given by $Ad_u\psi = u\psi u^{-1}$ for $u \in Spin_+(1,3)$ and $\psi \in \mathbb{R}_{1,3}$ and that $Ad_{-1} = Id = 1$, we see that this representation reduces to a representation of $SO_+(1,3)$ that is exactly ρ_c. Then, if the manifold $\mathcal{L}$ admits a spinor structure, we can also write

$$\mathcal{C}\ell(\mathcal{L}) = P_{Spin_+(1,3)}(\mathcal{L}) \times_{Ad} \mathbb{R}_{1,3}$$

This shows the difference between $\mathcal{C}\ell(\mathcal{L})$ and $\mathcal{C}\ell_{Spin_+(1,3)}(\mathcal{L})$.

We end this long (but necessary) introduction with the remark that explicit construction of the two-component a-spinors in $\mathbb{R}_{1,3}$ is presented in Appendix A. The description of the two-component a-spinor fields in $\mathcal{C}\ell_{Spin_+(1,3)}(\mathcal{L})$ is done in Appendix B.

2. GEROCH'S THEOREM

Our objective in this section is to prove the following result.

Theorem. Let $\mathcal{L}$ be as in Definition 0. Then a necessary and sufficient condition for $\mathcal{L}$ to admit a spinor structure is that the orthonormal frame bundle $F_0(\mathcal{L})$ have a global section.

Proof:

(i) We already observed that it is obvious from the definition of CSS (Definition 1) that if $F_0(\mathcal{L})$ is trivial, then a CSS always exists. We now prove the converse.

(ii) We start by observing that if $P_{Spin_+(1,3)}$ exists, then there also exist all possible spinor bundles and also the real and complex spinor bundles as defined in Section 1. In particular, $\mathcal{C}\ell_{Spin_+(1,3)}(\mathcal{L})$ exists and has a global section (Choquet-Bruhat *et al.*, 1982).

(iii) We now study the conditions for the existence of a two-component algebraic spinor in the real spinor-Clifford bundle $\mathcal{C}\ell_{Spin_+(1,3)}(\mathcal{L})$.

Given that any space-time admits a global timelike vector field e_0 (Sachs and Wu, 1977), we consider a local section h_α of $F_0(\mathcal{L})$ given by the local trivialization $\phi_\alpha: U_\alpha \times SO_+(1,3) \to \pi_c^{-1}(U_\alpha)$ $[U_\alpha \subset \mathcal{L}]$, where $h_\alpha(x)$ is a positive-oriented tetrad in $\pi_c^{-1}(x)$, $x \in \mathcal{L}$, where $e_0(x)$ is the first vector of the tetrad. Let h_β be another local section of $F_0(\mathcal{L})$ and let $e_0'(x)$ be the first vector of the tetrad in $\pi_c^{-1}(x)$. For $x \in U_\alpha \cap U_\beta$, there exists only one $g \in SO_+(1,3)$ such that $h_\beta(x) = h_\alpha(x)g$. As we are supposing the existence of a spinorial structure in $\mathscr{L}$, then to each $g \in SO_+(1,3)$ $\exists \pm u \in Spin_+(1,3)$ such that $w = uvu^{-1}$, $v, w \in T_x\mathscr{L} \equiv \mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$, which is the fiber over x of the real spinor bundle $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_\iota \mathbb{R}_{1,3}$ (we have identified as usual $\mathbb{R}^{1,3}$ with its image in $\mathbb{R}_{1,3}$). (See Appendix B.)

(iv) As shown in Appendix A, in $\mathbb{R}_{1,3|x}$ a two-component a-spinor field is determined by the idempotents $e_x = \frac{1}{2}[1 + e_3(x)e_0(x)]$ in ϕ_α and by $e'_x = \frac{1}{2}[1 + e_3(x)e_0(x)]$ in ϕ_β, where $e_3(x)$ and $e'_3(x)$ are, respectively, the third vectors of the tetrads h_α and h_β. In order for e_x and e'_x to define the same ideal in the fiber over x, it is necessary that

$$\mathbb{R}_{1,3}e_x = \mathbb{R}_{1,3}e'_x$$

This is obviously necessary for the spinor field to be uniquely defined in $U_\alpha \cap U_\beta$. Now,

$$\mathbb{R}_{1,3}e'_x = \mathbb{R}_{1,3}ue_xu^{-1} \Rightarrow \mathbb{R}_{1,3}e'_x = \mathbb{R}_{1,3}e_xu^{-1} \in \mathbb{R}_{1,3}e'_x$$

and since e_x is a primitive idempotent and e_xu^{-1} is also an element of a minimal right ideal, we get $e_xu^{-1} \in e_x\mathbb{R}_{1,3} \cap \mathbb{R}_{1,3} \simeq \mathbb{H}$, where $\mathbb{H}$ is the quaternion field (Figueiredo *et al.*, 1990; Porteous, 1981). Then $e_xu^{-1} = \psi \in \mathbb{H} \Rightarrow ue_xu^{-1} = u\psi e_x \Rightarrow ue_xu^{-1} = e_x$, since ue_xu^{-1} is also a primitive idempotent. We then have

$$e'_x = e_x \qquad \text{and} \qquad ue_x = e_xu$$

Since $e_x = \frac{1}{2}[1 + e_3(x)e_0(x)]$, we have

$$ue_3(x)u^{-1} = e_3(x), \qquad ue_0(x)u^{-1} = e_0(x)$$

This then implies a reduction of the group $SO_+(1,3)$ of $F_0(\mathscr{L})$ to $SO(2) \simeq S^1$.

(v) Observe that, as shown in Appendix B, we have an "a-spinorial metric" defined in $I_x = \mathbb{R}_{1,3}e_x$. Since we have a spinorial structure in $\mathscr{L}$ that preserves the ideal I_x (permitting the definition of an a-spinor field), it must preserve also the spinorial metric, i.e., we must have

$$e_1(x)e_0(x) = e'_1(x)e_0(x) \Rightarrow e'_1(x) = e_1(x)$$

This means that choosing this direction, it will be fixed for all $p \in \pi^{-1}(x) \subset F_0(\mathscr{L})$.

(vi) Finally, given that $\mathscr{L}$ is time and space-time oriented, we can choose $e_2(x)$ in a unique way in ϕ_α and then $e'_2(x) = e_2(x)$ for $e'_2(x)$ in ϕ_β. Then, the structural group of $F_0(\mathscr{L})$ reduces to the identity and the theorem is proved. ■

3. GRAF'S (1978) *e*-SPINOR STRUCTURE

In this section, unless specified, $\mathscr{L}$ is a general Lorentzian manifold of signature (p, q).

Definition 10 (Graf). An e-spinor structure is a nontrivial e-cross section of the Clifford bundle $\mathscr{C}\ell(\mathscr{L})$ such that e is an idempotent of global minimum rank.

The global e-rank is defined by

$$\text{rank } e = \max_{x \in \mathscr{L}}(\text{rank } e_x)$$

where rank e_x is the rank of the $\bigoplus \Lambda^d(\mathbb{R}^{p,q})$-morphism, $e\colon \psi \to \psi e$, where $\bigoplus \Lambda^d(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$ (Figueiredo *et al.*, 1990; Rodrigues *et al.*, 1989).

Given an e-spinor structure such that rank e is minimal for almost all $x \in \mathscr{L}$, we say that we have an "elementary e-spinor structure." Note that $e = 1$ defines a trivial e-spinor structure.

Definition 11 (Graf). Given two global idempotents e and e', we say that the e-spinor structure and the e'-spinor structure are equivalent if there exists an invertible cross section u with $e' = ueu'$ such that u induces an automorphism of $SO(p, q)$ [i.e., $u \in Spin_+(p, q)$].

Definition 12 (Graf). An e-spinor field on $\mathscr{C}\ell(\mathscr{L})$ corresponding to a given e-spin structure is a cross section ψ of $\mathscr{C}\ell(\mathscr{L})$ such that $\psi e = \psi$.

Graf (1978) states that the existence of an elementary e-spinor structure does not impose any global restrictions on $\mathscr{L}$. This is not true. Indeed, as every Lorentzian space-time $\mathscr{L}$ admits a timelike vector field e_0, then the global primitive idempotent $\frac{1}{2}(1 + e_0)$ defines naturally an "elementary e-spinor structure" and this structure does not imply any obstruction for $F_0(\mathscr{L})$. Then the $\breve{e} = \frac{1}{2}(1 + e_0)$ spinor structure always exists even if $w_2(\mathscr{L}) \neq 0$, since $\mathscr{C}\ell(\mathscr{L})$ always exists. When $(p, q) = 1, 3$, if $\psi \in \sec \mathscr{C}\ell(\mathscr{L})$ is such that $\psi\frac{1}{2}(1 + e_0) = \psi$, then ψ is a Dirac-spinor field, as proved in Figueiredo *et al.* (1990). However, if we want to have a cross section $\varphi \in \sec \mathscr{C}\ell(\mathscr{L})$ that is a Weyl e-spinor field (representing a two-component Weyl c-spinor field) equipped locally with the spinorial metric (see Appendix A), we need a global minimal idempotent $\frac{1}{2}(1 + e_0e_3)$, where e_0 is a global timelike field and e_3 is a global space field. This new e-spinor structure is not equivalent to the preceding one according to Definition 11, even if both idempotents are of global minimum rank equal to 8. Moreover, the existence of the cross section φ equipped with the spinorial metric implies that $F_0(\mathscr{L})$ is trivial.

At this point we must emphasize that Bugajska (1979) also found that the condition for the existence of a Weyl e-spinor field is that $F_0(\mathscr{L})$ must be trivial. However, she uses a complex Clifford bundle, i.e., the bundle $\bar{\mathscr{C}\ell}(\mathscr{L}) = P_{SO_+(1,3)} \times \mathbb{R}_{4,1}$, where $\mathbb{R}_{4,1}$ is the complexification of $\mathbb{R}_{1,3}$ (or $\mathbb{R}_{3,1}$) (Figueiredo *et al.*, 1990).

In this way the construction of Bugajska does not imply directly Geroch's theorem. To obtain Geroch's theorem, we must use $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ and study the conditions for the existence of a cross section that is a global Weyl a-spinor field as we did above.

The above discussions we believe make clear the distinction between the covariant spinor structure (Definition 1) and Graf's e-spinor structure (Definition 10).

Observation 1. Graf (1978) states that in a Lorentzian manifold of signature $p = 3$, $q = 1$, an everywhere timelike vector field e_0 with $g(e_0, e_0) = -1$ does not induce any nontrivial spinor structure within $P_{SO_+(3,1)} \times_{\rho_c} \mathbb{R}_{3,1}$, despite the fact that there are local e's with minimal rank equal to 4. This statement is, of course, true. However, we can avoid this difficulty simply by using the twisted-Clifford bundle, which is the construction of Atiyah *et al.* (1964).

In this construction we associate with the tangent space $\mathbb{R}_{p,q}$ the Clifford algebra $\mathbb{R}^{q,p}$ using the twisted Clifford group (Figueiredo *et al.*, 1990) by means of the twisted adjoint representation.

Using the Clifford bundle for $p = 3$, $q = 1$ as defined in this paper, the conditions for the existence of a Majorana spinor results from the form of the global idempotent (Figueiredo *et al.*, 1990) equal to the conditions found in Section 2.

4. CONCLUSIONS

In this paper we present a Clifford algebra approach to the spinor structure of space-time. We showed that, using the real spin-Clifford bundle $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$, a proof of Geroch's theorem follows once we determine the necessary and sufficient conditions for a global section of $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ to be a Weyl a-spinor field.

We also discuss Graf's notion of e-spinor structure, showing the relation of this concept with the covariant spinor structure. We show that, contrary to Graf's statement, there are e-spinor structures that imply obstructions to $F_0(\mathscr{L})$.

We also note that Bugajska's determination of the conditions for the existence of a Weyl spinor field is indeed the determination of the conditions for the existence of a Weyl e-spinor field as a global section $\varphi\check{e} = \varphi$ of the complex Clifford bundle $\bar{\mathscr{C}\ell}(\mathscr{L}) = P_{SO_+(1,3)} \times_{\rho_c} \mathbb{R}_{4,1}$. Then, Bugajska's construction does not imply a proof of Geroch's theorem. She instead obtains that from her construction a definition of spinor structure follows that is identical to the covariant spinor structure of Definition 1.

APPENDIX A

Let $\mathbb{C}^2$ and $^*\mathbb{C}^2$ be two copies of a two-dimensional complex vector space.

A1. The contrariant (undotted) two-component spinors are the elements of the vector space $\mathbb{C}^2$ where there is defined a "c-spinorial metric"

$$\beta\colon\ \mathbb{C}^2\times\mathbb{C}^2\to\mathbb{C},\qquad \beta(\psi,\phi)=-\beta(\phi,\psi)$$

The c-spinorial metric β is clearly invariant under the action of the group $SL(2,\mathbb{C})$, i.e., given $u\in SL(2,\mathbb{C})$, we have

$$\psi\to u\psi,\qquad \phi\to u\phi,\qquad \text{and}\qquad \beta(\psi,\phi)=\beta(u\psi,u\phi)$$

In the canonical basis of $\mathbb{C}^2$ we have the following matrix representation:

$$\beta(\psi,\phi)=\psi^t\mathscr{C}\phi;\qquad \phi=\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix};\qquad \psi^t=(\psi^1,\psi^2);$$

$$\mathscr{C}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\qquad \psi^i,\phi^i=1,2\in\mathbb{C}$$

The spinorial metric β determines a canonical isomorphism between $\mathbb{C}^2$ and $^*\mathbb{C}^2$, given by $\beta\colon\mathbb{C}^2\to{}^*\mathbb{C}^2$; $\psi\to\beta(\psi,\cdot)={}^*\psi$.

A2. The "covariant" (undotted) two-component spinors are the elements of $^*\mathbb{C}^2$, where there is defined a "c-spinorial metric" $^*\beta\colon{}^*\mathbb{C}^2\times{}^*\mathbb{C}^2\to\mathbb{C}$; $^*\beta(^*\psi,{}^*\phi)=\beta(\psi,\phi)$.

The "c-spinorial metric" $^*\beta$ is clearly invariant under the actions of $SL(2,\mathbb{C})$ i.e., $^*\psi\to{}^*\psi u^{-1}$, $^*\phi\to{}^*\phi u^{-1}$ and $^*\beta(^*\psi u^{-1},{}^*\phi u^{-1})={}^*\beta(^*\psi,{}^*\phi)$.

In the canonical basis of $^*\mathbb{C}^2$, we have $^*\psi=(\psi_1,\psi_2)\equiv(\psi^2,-\psi^1)$

Observation. In Figueiredo *et al.* (1990) we used the notations $\overset{\Delta}{\underset{\bullet}{\mathbb{C}}}{}^2$ and $\overset{\Delta}{\psi}{}^2$ for $^*\mathbb{C}^2$ and $^*\psi$ and called covariant spinor (c-spinors) objects of complex vector spaces which transform with a well-defined rule under the action of a given spin group. In this sense both ψ and $^*\psi$ are c-spinors.

Let $\mathbb{R}_{1,3}$ be the space-time algebra [see Figueiredo *et al.* (1990) for details of the notation], i.e., the real Clifford algebra generated by the vectors E_μ, $\mu=0,1,2,3\in\mathbb{R}^{1,3}\hookrightarrow\mathbb{R}_{1,3}$ such that

$$E_\mu E_\nu+E_\nu E_\mu=2\eta_{\mu\nu}$$

$$\eta_{00}=1;\qquad \eta_{ii}=1,\quad i=1,2,3;\qquad \eta_{\mu\nu}=0,\quad \mu\neq\nu$$

Let $\mathbb{R}_{1,3}^{+} \simeq \mathbb{R}_{3,0}$ be the even subalgebra of $\mathbb{R}_{1,3}$. Then we can easily show (Figueiredo *et al.*, 1990) that $E = \frac{1}{2}(1 + E_3 E_0)$ is simultaneously a primitive idempotent of both $\mathbb{R}_{1,3}$ and $\mathbb{R}_{3,0}$.

A3. The contravariant (undotted) two-component algebraic spinors are the elements of the minimal left deal $I = \mathbb{R}_{1,3}^{+} E$.

The spinor basis in I is $\{S_A, A = 1, 2;\ S_1 = E,\ S_2 = E_1 E_0 E\}$ and we have the following representation of ψ in the spinorial basis:

$$I \ni \psi = E\psi^1 + E_1 E_0 E \psi^2 \simeq \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} \in \mathbb{C}(2) \qquad \psi^A, \quad A = 1, 2 \in \mathbb{C}$$

Consider the space $I^{\sim} = (\mathbb{R}_{1,3}^{+} E)^{\sim}$, where $\sim$ is the composition of the main automorphism and the main antiautomorphism in $\mathbb{R}_{1,3}$ (Figueiredo *et al.*, 1990). Then, give $\psi \in I$, and being $\bar{E} = 1 - E$, we have

$$I^{\sim} \ni \psi^{\sim} = E\psi^1 - EE_1 E_0 \psi^2 \simeq \begin{pmatrix} 0 & 0 \\ -\psi^2 & \psi^1 \end{pmatrix} \in \mathbb{C}(2)$$

A4. The covariant (undotted) two-component algebraic spinors are the elements ${}^*\psi = E_1 E_0 \psi^{\sim} \in {}^*I = E\mathbb{R}_{1,3}^{+}$,

$${}^*I \ni {}^*\psi = E_1 E_0 \psi^{\sim} = EE_1 E_0 \psi^1 - E\psi^2 \simeq \begin{pmatrix} -\psi^2 & \psi^1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_2 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}(2) \simeq {}^*\mathbb{C}(2)$$

In $I = \mathbb{R}_{1,3} E$ we define an a-spinorial metric $\hat{\beta}$ that mimics the c-spinorial metric β defined in A1.

A5. The a-spinorial metric $\hat{\beta}$ is

$$\hat{\beta}\colon \quad I \times I \to \mathbb{C} \quad \text{by} \quad \hat{\beta}(\psi, \phi) = -\hat{\beta}(\phi, \psi) = 2\langle E_0 E_1 \psi \phi \rangle_0$$

where $\langle \cdot \rangle_0$ means the scalar part of the Clifford number (Figueiredo *et al.*, 1990).

Note that $\hat{\beta}$ is $Spin_+(1,3) \simeq SL(2, \mathbb{C})$ invariant in the following sense. Let $u \in Spin_+(1,3)$. Then $u^{\sim} u = 1$ (Figueiredo *et al.*, 1990). It follows that $\hat{\beta}(u\psi, u\phi) = \hat{\beta}(\psi, \phi)$.

For brevity in what follows we call the elements of I and ${}^*I$ a-spinors.

APPENDIX B: $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L}) = P_{Spin_+(1,3)}(\mathscr{L}) \times_\iota \mathbb{R}_{1,3}$

B1. The real spinor bundle $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ is a "principal $\mathbb{R}_{1,3}$ bundle," in the sense that we have a right action of $\mathbb{R}_{1,3}$ defined in each fiber $\pi^{-1}(x) \simeq \mathbb{R}_{1,3}$. We describe explicitly this action in the following. $P_{Spin_+(1,3)}(\mathscr{L}) \to \mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ is a natural embedding, as a consequence of the embedding $Spin_+(1,3) = \mathbb{R}_{1,3}^{+} \subset \mathbb{R}_{1,3}$ (Figueiredo *et al.*, 1990).

B2. Let $F_0(\mathscr{L})$ be the bundle of orthonormal frames $[F_0(\mathscr{L}) = P_{SO_+(1,3)}(\mathscr{L})]$ and let $\{U_\alpha\}$ be a covering of $\mathscr{L}$. Let $h_\alpha: U_\alpha \to \pi^{-1}(U_\alpha)$ be a local section given by the local trivialization $\mu_\alpha: \pi^{-1}(U) \to U_\alpha \times SO_+(1,3)$. Then $h_\alpha(x)$ is an oriented tetrad in $\pi^{-1}(x)$. We put $h_\alpha(x) = (e_0(x), e_1(x), e_2(x), e_3(x))$. Let $h_\beta: U_\beta \to \pi^{-1}(U_\alpha)$ be another local section. We put $h_\beta(x) = (e'_0(x), e'_1(x), e'_2(x), e'_3(x))$. Now, suppose $U_\alpha \cap U_\beta \neq \emptyset$. Then there exists a unique $g \in SO_+(1,3)$ such that $h_\beta(x) = h_\alpha(x)g$. But since we are supposing the existence of a spinorial structure in $\mathscr{L}$, there exists $\pm u \in Spin_+(1,3)$ such that $\pm uv(\pm u)^{-1} = w$, with $v, w \in T_x\mathscr{L} = \mathbb{R}^{1,3}$. Then, $h_\beta(x) = uh_\alpha(x)u^{-1}$ imagining $T_x\mathscr{L} = \mathbb{R}^{1,3}$ canonically embedded in $\mathbb{R}_{1,3|x} \simeq \mathbb{R}_{1,3}$, the fiber over x in $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$.

B3. Let $\phi_\alpha: \pi_c^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}_{1,3}$ and $\phi_\beta: \pi_c^{-1}(U) \to U_\beta \times \mathbb{R}_{1,3}$ be two local trivializations of $\mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$. Let $\Psi: \mathscr{L} \supset U \to \pi_c^{-1}(U)$ be a local section, given by

$$\Psi(x) = (x, \psi(x))$$

The homeomorphisms ϕ_α and ϕ_β have the form

$$\phi_\alpha(\Psi(x)) = (\pi_c(\Psi(x)), \qquad \hat{\phi}_\alpha(\psi(x))) = (x, \psi_\alpha(x))$$

and

$$\psi_\alpha(x) = \hat{\phi}_\alpha \circ \hat{\phi}_\beta^{-1} \psi_\beta(x) = u\psi_\beta(x)$$
$$u = \hat{\phi}_\alpha \circ \hat{\phi}_\beta^{-1} \in Spin_+(1,3); \qquad \psi_\alpha, \psi_\beta \in \mathbb{R}_{1,3}$$

Now, associated with $h_\alpha(x)$ we have the primitive idempotent $e_x = \frac{1}{2}[1 + e_3(x)e_0(x)]$, which defines the ideal $I_x = \mathbb{R}^+_{1,3}e|_x$. The canonical spinorial basis of I_x is $\{s_A(x), A = 1, 2\}$. Associated with $h_\beta(x)$ we have the primitive $e'_x = \frac{1}{2}[1 + e'_3(x)e'_0(x)]$, which defines the ideal $I'_x = \mathbb{R}^+_{1,3}e'_x$ and in general $I_x \neq I'_x$. The canonical basis of I'_x is $\{s'_A(x), A = 1, 2\}$.

We put $\hat{\phi}_\alpha(e_\mu) = E_\mu$, which is by definition the canonical basis of $\mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$. We put $\hat{\phi}_\alpha(e'_\mu) = E'_\mu$,

$$E'_\mu = uE_\mu\mu^{-1} = L^\nu_\mu e_\nu, \qquad L^\nu_\mu \in SO_+(1,3)$$

Also $\phi_\alpha(s_A) = S^\alpha_A$, the canonical basis of $I = \mathbb{R}_{1,3}E$, $E = \frac{1}{2}(1 + E_3E_0)$, and $\phi_\beta(s'_A) = S'^\beta_A$ where S'^β_A is the canonical basis of $I' = \mathbb{R}_{1,3}E'$, $E' = \frac{1}{2}(1 + E'_3E'_0)$.

Observe that

$$\phi_\beta(s_A) = S^\beta_A, \qquad S^\alpha_A = uS^\beta_A \equiv \Lambda(u)^B_A S^\beta_B \in I$$
$$\phi_\alpha(s'_A) = S'^\alpha_A, \qquad S'^\alpha_A = uS'^\alpha_A = \Lambda(u)^B_A S'^\beta_B \in I'$$

but

$$S'^\beta_A = uS^\alpha_A u^{-1}$$

B4. If $\Psi\colon U_\alpha \to \mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ is such that $\psi(x)e_x = \psi(x)$ for all $x \in U_\alpha$, we call Ψ a local "a-spinor field."

B5. Here we consider the right action $\mathbb{R}_u$. Let $\Psi\colon U_\alpha \to \mathscr{C}\ell_{Spin_+(1,3)}(\mathscr{L})$ be a section. We define the right action of $u \in Spin_+(1,3) \simeq \mathbb{R}^+_{1,3}$ by

$$R_u\colon \quad \mathbb{R}_{1,3|x} \to \mathbb{R}_{1,3|x}$$
$$R_u\psi(x) = \psi(x)u^{-1}$$

Now, since for an a-spinor field $\psi(x) = \psi(x)e_x$, we have

$$\phi'(x) = \psi(x)u^{-1} = \psi(x)e_x u^{-1} = \psi(x)u^{-1}\, e'_x = \phi'(x)e'_x; \qquad e'_x = ue_x u^{-1}$$

This means that the right action R_u is not ideal preserving, but as R_u acts freely on $\mathbb{R}_{1,3|x}$, we have that

$$R_u|_{\mathbb{R}_{1,3}e_x}\colon \quad I_x \to I'_x \text{ is an isomorphism}$$

Note that

$$\psi u^{-1} = \left(\sum_A \psi^A_\alpha s_A\right) u^{-1} = \sum_A \psi^A_\alpha u^{-1} s_A u^{-1} \left(\sum_A \psi^A_\alpha s'_A\right)$$

Then

$$\phi_\beta(\psi u^{-1}) = \phi_\beta\left(u^{-1} \sum_A \psi^A_\alpha s'_A\right)$$
$$= \sum \Lambda^{-1}(u)^A_B \psi^B_\alpha s'^\beta_A$$
$$\phi'^A_\beta = \Lambda^{-1}(u)^A_B \psi^B_\alpha = \Lambda^{-1}(u)^A_B \Lambda(u)^B_C \psi^C_\beta = \psi^A_\beta$$

This last equation means that the transformed spinor $\phi' = \psi u^{-1} \in I'_x$ has in the basis s'_A the same components that ψ has in the basis $\Lambda^{-1}(u)^B_A s_A$ in I_x.

B6. Here we consider the spinorial metric. We define the metrics $\hat{\beta}$, $\hat{\beta}'$ by

$$\hat{\beta}\colon \quad I_x \times I_x \to \mathbb{C}, \qquad \hat{\beta}(\psi, \phi) = 2\langle E_1 E_0 \psi\phi\rangle_0$$
$$\hat{\beta}'\colon \quad I'_x \times I'_x \to \mathbb{C}, \qquad \hat{\beta}'(\psi', \phi') = 2\langle E'_1 E'_0 \psi'\phi'\rangle_0$$

We have the identities: (i) $\hat{\beta}(\psi, \phi) = \hat{\beta}(u\psi, u\phi)$; (ii) $\hat{\beta}'(\psi'; \phi') = \hat{\beta}'(\psi u^{-1}, \phi u^{-1}) = \hat{\beta}(\psi, \phi)$; (iii) $\hat{\beta}'(u\psi u^{-1}, u\phi u^{-1}) = \hat{\beta}(\psi, \phi)$.

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